Toward a descent theoretic formulation for organization and emergence -An initial object sheaf hypothesis and its consequence

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- An initial object sheaf α hypothesis and its consequence

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Abstract. Let α be an initial object (let ω be a terminal object) of a temporal topos $S^{\wedge}$; by definition for an arbitrary object m of $S^{\wedge}$ there exists a uniquely determined morphism $\delta_m: \alpha \to m$ ($\sigma_m: m \to \omega$). Sheaf α, an initial object of $S^{\wedge}$, plays a crucial and functional role for our cognitive study in this paper, where a terminal object ω plays an essential role for the temporal topos theoretic methods for the fundamental aspects of quantum physics and quantum gravity as in [5], [6], [7], [8], [9]. Via the methods of descent theory e.g., as in [1], sheaf α can give explicit categorical formulations for cognitive concepts and phenomena in what will follow. Under the hypothesis on the existence of sheaf α as an initial object of t-topos, the functionality of the role among fundamental phenomena will be exposed. As an example, a sheaf (stack)- categorical formulation is given for the mechanism for the emergence from locally trivial individual ant's behavior of a leaf-collecting ant colony to the global and organized consequence as a colony. A fundamental principle of this paper is that a prestack is a stack if a system is self-organized or local objects may be patched together to become a global object.

Keywords: Topos, Sheaf, Category, Decent theory, Self-organization

1. Introduction

Unifying methods from the local to the global come from the descent theory. The descent theory was originally developed in algebraic geometry by Alexander Grothendieck, in terms of sheaves and stacks. The category of sheaves over a site S is said to be a Grothendieck topos (see [2], [3], [4]). We have introduced the notion of temporal topos $S^{\wedge}$. In what will follow, we abbreviate it as t-topos. Namely, t-topos $S^{\wedge}$ is the category of presheaves over a t-site S with the t-topos theoretic restrictions on the site S as defined in [5], [6], [7], [8], and [9]. As an application to physics, especially quantum physics, special and general relativity and quantum gravity, a terminal object ω as space-time sheaf plays an important role as in [5, 6, 7, 8, 9, 10].

One of the fundamental categorical mottos is to emphasize the importance of morphisms rather than objects of a category. For an initial object α in a category, an arbitrary object m may be identified with the uniquely determined morphism δ_m from an initial object α to m. The transition from the categorical to the topos (sheaf) theoretic is the t-topos approach which is temporally controlled (parameterized) by t-site objects.

Notice the duality between a terminal object ω and an initial object α for the physical and cognitive aspects of t-topos, respectively.

Our fundamental view can be schematically expressed in the following diagram: for a terminal object ω:

$$\begin{array}{ccccc}
\text{elt. particles} & & \overset{a}{\underset{\text{physics}}{\rightleftarrows}} & & \text{matter} \\
 & \searrow & & \swarrow & \\
\uparrow\downarrow^{d} & & \omega & & \uparrow_{\text{bio-phy.}}\downarrow^{b} \\
 & \nearrow & & \nwarrow & \\
\text{consc.} & & \overset{\text{bio-med}}{\underset{c}{\rightleftarrows}} & & \text{life}
\end{array}$$

and for an initial object α:

$$\begin{array}{ccccc}
\text{elt. particles} & & \overset{a}{\underset{\text{physics}}{\rightleftarrows}} & & \text{matter} \\
 & \nwarrow & & \nearrow & \\
\uparrow\downarrow^{d} & & \alpha & & \uparrow_{\text{bio-phy.}}\downarrow^{b} \\
 & \swarrow & & \searrow & \\
\text{consc.} & & \overset{\text{bio-med}}{\underset{c}{\rightleftarrows}} & & \text{life}
\end{array}$$

The direction of a, b, and c in the above diagrams correspond to the organization-emergence from the small to the large. Note that directions a, b, c, d in the above diagram corresponds to Artificial Intelligence in a more general sense. This is because all those directions a, b, c, d share the following common theme; locally trivial discrete data are organized, via the notion of a sheaf or a stack, into a non-trivial global object. For example, locally trivial digital data are pasted (organized) together so that global outcomes like ‘decisions’ or ‘choices’ may be approximated to human activities. Those directions a, b, c, and d require the notion of descent in the following sense: obtaining a global object from local information and pasting their data.

It may be appropriate to make a few comments on the philosophical aspects that are relevant to the notion of sheaf α. Plato’s idea of an alive cosmos with mind and G. Leibniz’s concept of monads as sentient particles are relevant to the categorical role of sheaf α. To be explicit, for a fundamental presheaf m, there exists a unique morphism δ_m from an initial object α to m. This morphism corresponds to a fundamental monad in Leibniz’s sense. See [5] for the definition of a fundamental presheaf. Notice also that the approaches taken by B. Spinoza, G. W. F. Hegel, and F. Schelling with the knowledge of Buddhism, considering the source of universal consciousness as Brahman-atman of No-Self, seem to be closer to the ancient Greek philosophies of Plato. It might be a worthwhile effort for the reader with philosophical interest to provide explicit categorical foundations and formulations to those philosophers' ideas in terms of sheaf α.

After a brief review on t-topos theoretic methods in section 2, in section 3 on Memory, we aim at how the past thought can be lifted to the present thought via the composition of a morphism from an initial object α with a non-canonical morphism from the past state to the present state. In section 4, this is an outline of the forthcoming paper of descent theoretic formulation of psychological phenomenon 'discovery'. Section 5 is our main result of this paper in terms of the notion of a stack for an ant-colonial behavior as an organized whole consisting of the individual ants.

2. Terminal Object ω and Initial Object α of Temporal Topos

Before we investigate our approach to this paper, we will provide a short summary of the methods of t-topos as follows.

In terms of the *t-topos* theory developed in [5, 6, 7, 8, 9], an ur-particle state of a presheaf m over a generalized time period V, which corresponds to collapse of the wave function, is noted as the reified state $m(V)$ of presheaf m at V. Namely, this state-vector reduction is the jump from the continuous and deterministic to the discontinuous and probabilistic in the quantum mechanical sense. For the usual time linearity $t < s$, e.g., t corresponds to the present time, and s corresponds to a future time, we say that there corresponds a t-linear morphism in the t-site

$$g: V \to U \tag{1}$$

where t and s correspond to V and U, expressed as $t \sim \tau(V)$ and $s \sim \tau(U)$ for the t-topos time-sheaf τ. Then the following morphism is functorially induced by (as a contravariant-ness of functor m) presheaf m in $S^{\wedge}$

$$m(V) \overset{m(g)}{\longleftarrow} m(U). \tag{2}$$

Then, for a t-linear $V \to W$ a measurement or observation morphism s_W^V by an observer P over W is expressed as the vertical morphism in the following diagram

$$\begin{array}{ccc} m(V) & \overset{m(g)}{\longleftarrow} & m(U) \\ \downarrow s_W^V & & \\ P(W) & & \end{array} \tag{3}$$

indicating measurement of the *ur*-state of $m(V)$ by P over W provides the information of the future *ur*-particle state of m over U by composing the morphisms $m(g)$ with s_w^V. Namely, we have the composed morphism

$$s_W^V \circ m(g): m(U) \to P(W). \tag{4}$$

Notice that the above formulation is relativistic in the sense that s_W^V is a non-canonical (non-functorial) morphism associated with a t-linear morphism $V \to W$ in the t-site S. Namely, two *ur*-particle states of m and P, i.e., $m(V)$ and $P(W)$, are mutually in the same t-light cone. For the details for the relativistic aspect, see [5, 7, 9]. Thus, t-topos methods have shown the usefulness in the field of foundations of quantum physics as in [10], and quantum gravity (see [5, 7, 9]).

As a crucial and also fundamental diagram as mentioned in Introduction, and even as an *ultimate reality* based on the philosophical views also mentioned in the Introduction, we have the following diagram:

$$\begin{array}{ccc} \alpha & \overset{\delta_\omega=\sigma_\alpha}{\longrightarrow} & \omega \\ & \searrow^{\delta_m} \ \nearrow_{\sigma_m} & \\ & m & \end{array} \tag{5}$$

All the morphisms in diagram (5) are uniquely determined by the very definitions of initial and terminal objects α and ω, respectively.

Notes: (1) As mentioned earlier, the uniqueness of morphism δ_m for a given object m of t-topos $S^\wedge$ tells us the following categorical motto. We should pay attention to the morphism δ_m rather than object m. In terms of consciousness, it seems more appropriate to regard δ_m^V as the consciousness of m over the generalized time period V, which is an object of the t-site S. For the physical aspect of the *ur*-particle state $m(V)$ over V, one should emphasize the unique morphism $\sigma_m^V : m(V) \to \omega(V)$ whose quantum and relativistic applications are described in [5, 6, 7, 8].

(2) Notice that such a consciousness δ_m^V of m is parameterized by an object V as a generalized time period of t-site S. Namely, the consciousness of m over the generalized time period V, or in terms of time-sheaf $\tau(V)$, seems most appropriate to be defined as $\delta_m^V: \alpha(V) \to m(V)$. Self-awareness can be expressed as an identity morphism $1_m: m(V) \to m(V)$. Or rather, we have the commutative diagram:

$$\begin{array}{ccc} \alpha(V) & = & \alpha(V) \\ \downarrow \delta_m^V & & \downarrow \delta_m^V \\ m(V) & \longrightarrow & m(V) \end{array}$$

where the commutativity in this case simply indicates 1_m composed with δ_m^V equals by definition δ_m^V composed with 1_m.

Hypothesis on α: sheaf α and all the objects m in t-topos $S^\wedge$ are t-entangled the sense of t-topos theory. That is, the pair of m and α acts as one presheaf as defined in [5, 6, 7, 8, 9].

This t-entanglement hypothesis on α implies the following. Let $f: V \to U$ be a morphism in the t-site S. Then we have the commutative diagram as follows:

$$\begin{array}{ccc} \alpha(U) & = & \alpha(V) \\ \downarrow \delta_m^U & & \downarrow \delta_m^V \\ m(U) & \overset{m(f)}{\longrightarrow} & m(V) \end{array} \tag{6}$$

This commutative diagram (6) is crucial for applications to biological aspects; even though a brain controls our body, it still does not explain, for example, how simple movement e.g., standing up or walking triggers all the nerves and muscles simultaneously in such an organized way. The commutativity in (6) indicates non-canonical (i.e., non-functorial) behaviors in the global level of a living entity. The process from local data, e.g., individual nerves to a global action of a body, requires the descent theory.

Remarks. (1) The initial and terminal objects are constant functors assigning any object of S to initial and final objects, respectively. That is, for example, $\alpha(V)$ is an initial object in the target category for all reifiable objects V in the t-site S. Consequently, for a t-linear morphism $f: V \to U$ in the t-site S, we get an initial object $\alpha(U) = \alpha(V)$ in the above commutative diagram (6) for arbitrary objects U and V of the t-site. Hence we can write $\alpha(U) = \alpha(\cdot)$, or even simply α, for any object U in the t-site S.

(2) Such an initial object sheaf α of t-topos can be called a universal consciousness to be compatible with traditional philosophical terminologies used by Plato, Leibniz, Spinoza, Hegel, ... as mentioned in the Introduction.

(3) An arbitrary object m of t-topos is uniquely associated with δ_m , as in (5) and (6), which is the uniquely determined morphism from universal consciousness α to m. Using the notation in (1), for any object V of a t-site, there exists a unique morphism $\delta_m : \alpha(V) = \alpha(\cdot) \to m(V)$.

3. Memory

In t-topos theoretic method, a morphism $V \to W$ is said to be t-linear, when the state determined by V precedes the state over W, e.g., if the past event corresponds to V, then the present event corresponds to W.

As an observation or a measurement by an observer p for an observed m is a non-canonical (non-functorial) action (morphism) in the sense of t-topos. Namely, when p measures m, then there is the following non-canonical morphism

$$s_W^V : m(V) \longrightarrow p(W). \tag{7}$$

Note that

$$g : V \longrightarrow W \tag{8}$$

is a t-linear morphism in the t-site S, in the sense of t-topos theory, i.e., $t \sim \tau(V)$ precedes $s \sim \tau(W)$ in the usual sense of linearly ordered time. See [5] for the definition and related topics of a t-linear morphism.

On the other hand, for a t-linear morphism g in (8), there is induced the canonical morphism $p(g)$ by the contravariant functor p

$$p(W) \overset{p(g)}{\longrightarrow} p(V).$$

A non-canonical memory-morphism is defined as the reversed direction of the above, i.e.,

$$p(V) \overset{p_W^V}{\longrightarrow} p(W), \tag{9}$$

satisfying the following commutativity. For initial object α in the t-topos, we obtain the following commutative diagram:

$$\begin{array}{lcl} \alpha(V) & & \\ \downarrow \delta_p(V) & \searrow^{\delta_p(W)} & \\ p(V) & \overset{p_W^V}{\longrightarrow} & p(W) \end{array} \tag{10}$$

To be precise, the memory at the state $p(W)$ of the past V is formulated as $\delta_p(W)$ in (10), namely the composition of the above morphisms

$$p_W^V \circ \delta_p(V) : \alpha(V) \longrightarrow p(W).$$

4. Psychological aspect of mathematical discovery in terms of t-topos theory

Besides my own experiences and having interviewed several mathematicians, discovery moments in mathematics may be characterized as follows. Such moments come suddenly and unexpectedly without much apparent effort solving problems (or discovering crucial new ideas); however, it is usually after a long period of devoted efforts and thoughts toward the solutions.

Let B be the presheaf associated with a brain or a conscious entity. Let $B(V)$ be an *ur*- particle state during the generalized time interval V. For a t-linear morphism $f: V \to W$, we have

$$\begin{array}{ccc} \alpha(V) & = & \alpha(W) \\ \downarrow^{\delta_B^V} & & \downarrow^{\delta_B^W} \\ B(V) & \overset{B(f)}{\longleftarrow} & B(W) \end{array} . \tag{11}$$

Suppose that a discovery is made over W or in terms of time-sheaf, at $\tau(W)$. The commutativity of the above diagram indicates that over the preceding generalized time V, B gets the information from the discovered object $\alpha(W)$ via δ_B^W composed with $B(f)$ which would indicate a certain amount of prediction of the discovery made over W. However, when such an unexpected discovery is made, the conscious entity B ceases to be functorial in the sense that $B(f)$ does not exist so that the above diagram (11) becomes commutative.

5. Ant colony, an organized whole as a prototype of descent theory

In this section, as a basic principle of t-topos theory as done in [5, 6, 7, 8, 9], the transition from locally trivial objects (in this case an individual ant's behavior) to a unified global whole (as a unit of a colony of ants) will be described in terms of categorical-sheaf theoretic notions. The theory of descent in [1] will play a decisive role for our approach via the t-topos. It should be pointed out that an individual ant is represented by a presheaf, not a sheaf.

Let

$$B = \coprod b^i$$

be the coproduct of presheaves $\{b^i\}$ associated with the totality of presheaves representing the entire ants in an ant-colony, defined over a t-site S, where i is an element of the index set $i = 1,2,\ldots.N$. Namely, N is the number of ants in the colony. Note that N varies and depends upon time. For each i, let us assume that each $b^i$is reified over a t-site object U. We have a unique morphism from α to an arbitrary object of t-topos $S^\wedge$ by the definition of an initial object, i.e.,

$$\delta^i = \delta_b^i : \alpha \longrightarrow b^i.$$

Let $C(\{b^i \to B\})$ be the category whose objects are the collections of morphisms with descent data. That is, a collection of morphisms $\delta^i = \delta_b^i : \alpha \longrightarrow b^i$ so that $g_{ij}: \pi_2^* \delta^j \to \pi_1^* \delta^i$ is an isomorphism where π_1 and π_2 are the following vertical and horizontal projection morphisms in

$$\begin{array}{ccc} b_i \times b_j = b_{ij} & \longrightarrow & b_j \\ \downarrow & & \downarrow \\ b_i & \longrightarrow & B \end{array}$$

in the t-topos $S^\wedge$ of presheaves $\{b_i\}$.

In the above case, an object is a natural transformation δ^i from an initial object α to a presheaf (a contravariant functor) b_i in t-topos $S^\wedge$, i.e., an object δ^i is a morphism in a functor category $S^\wedge$.

Our goal is to obtain a global morphism $\delta^B : \alpha \to B$ from local data $\{\delta^i = \delta^i_b\}$ so that the restriction of such a global morphism δ^B to each b^icoincides with each morphism δ^i. Then the following *cocycle condition* (12) is satisfied, via *gluing* (or *transition*) *isomorphisms* $g_{ij} : \pi_2^* \delta^j \to \pi_1^* \delta^i$,

$$\pi_{13}^* g_{ik} = \pi_{12}^* g_{ij} \pi_{23}^* g_{jk} : \pi_3^* \delta^k \to \pi_1^* \delta^i. \tag{12}$$

Namely, the transition morphism g_{ij} is an isomorphism from δ^i to δ^j over the "intersection" $b_i \times b_j = b_{ij}$. Then the global $\delta^B : \alpha \to B$ in the following diagram

$$\begin{array}{ll} \alpha & \\ \downarrow \delta^B & \searrow^{\delta^i} \\ B = \coprod b^i & \longleftarrow b^i \end{array} \tag{13}$$

is determined.

In the following diagram

$$\begin{array}{ccc} \alpha & \longrightarrow & \alpha \\ \downarrow \delta^i & & \downarrow \delta^j \\ b^i & \longrightarrow & b^j \end{array} \tag{14}$$

notice that the above isomorphism $g_{ij} : \pi_2^* \delta^j \to \pi_1^* \delta^i$ glues δ^i and δ^j together over $b_i \times b_j = b_{ij}$ to obtain the global $\delta^B : \alpha \to B$, where the horizontal $\alpha \to \alpha$ in diagram (14) is an identity morphism 1_α on an initial object α of the t-topos $S^\wedge$.

Consequently, the functor from $C(S^\wedge)$ (the category of ant-colonies in this case) to the category $C(\{b^i \to B\})$ with descent data, assigning δ^B to the associated descent data by the pullbacks of the morphisms $b^i \to B$, is an equivalence of categories. Namely, the existence of the global object δ^B from the local descent data $(\{\delta^i\}, \{g_{ij} : \pi_2^* \delta^j \to \pi_1^* \delta^i\})$ asserts that C is a stack over the t-topos $S^\wedge$. Note also that our descent theoretic formulation is independent of (or irrelevant to) a t-site but dependent on the t-topos $S^\wedge$.

6. Peroration

As for topics in Section 4, H. Poincare, J. Hadamard, K. Oka, H. Weyl, etc. described their experiences. See for example [10]. The main result of this paper is Section 5: Ant colony, an organized whole as a prototype of descent theory. As a future investigation, such a fundamental question as "what way the permutations of DNA sequences are capable of determining the actual corresponding physical outcomes" needs to be considered. Embryology is another related and profound problem. We seek for explicit sheaf-stack theoretic formulations for such phenomena as our future projects.

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